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Source: *The Scandinavian Journal of Economics*, Vol. 107, No. 3 (Sep., 2005), pp. 547-561
Published by: Wiley on behalf of The Scandinavian Journal of Economics
Stable URL: https://www.jstor.org/stable/3440972
Accessed: 29-08-2019 18:49 UTC

Scand. J. of Economics 107(3), 547–561, 2005
DOI: 10.1111/j.1467-9442.2005.00422.x

Intertemporal Equity and Hartwick's Rule in an Exhaustible Resource Model*

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Abstract

In a standard exhaustible resource model, it is known that if, along a competitive path, investment in the augmentable capital good equals the rents on the exhaustible resource (known as Hartwick's rule), then the path is equitable in the sense that the consumption level is constant over time. In this paper, we show the converse of this result: if a competitive path is equitable, then it must satisfy Hartwick's rule.

Keywords: Intertemporal equity; exhaustible resource; Hartwick's rule; Hotelling's rule
JEL classification: *D*90; *O*11; *O*41; *Q*32

I. Introduction

The purpose of this paper is to show that equitable paths in an infinite-horizon exhaustible resource model can be completely characterized in terms of *Hartwick's rule*: invest the rent from the exhaustible resource used at each date in the net accumulation of the produced capital good.

This area of study originates with a paper by Solow (1974), who analyzed a capital accumulation model, with Cobb–Douglas technology, in the presence of an exhaustible resource. He was interested in the possibility of sustainable consumption levels in this context and, eschewing the use of the

* Research on this paper was started when Dasgupta was visiting Cornell University, whose research facilities, as well as sabbatical leave support from Dalhousie University, are gratefully acknowledged. The current version has benefited from valuable comments by two referees of this journal.

traditional discounted integral of utilities as a measure of welfare, concentrated attention on the welfare of the least well-off generation. His choice of the objective was of the Rawlsian maximin type, seeking to maximize the least consumption level that can be assured along growth paths from given initial resources. Such a path is efficient as well as equitable, where equity in this context means that the path maintains a constant consumption level at all dates. Subsequently, Hartwick (1977) made the interesting observation that a competitive path, which follows the simple rule of thumb of investing the rents from the exhaustible resources used at each date in the net accumulation of produced capital goods, is equitable. We refer to this investment rule as Hartwick's rule. As Solow (1986) has observed, this is an intuitively appealing investment rule of maintaining the consumption potential of society, in a generalized sense, by replacing exhaustible resource stocks, which are used up, with produced capital goods of equal value.

It turns out that Hartwick's rule has significance in a wider class of models than the special context in which it arose initially. In particular, Dixit, Hammond and Hoel (1980) recognized that Hartwick's rule is really a statement that the valuation of net investment (including the dis-investment in the exhaustible resource) is zero at each date. They then proceeded to show, in a general model of accumulation involving heterogeneous capital goods (which could include various non-renewable resource stocks), that if the valuation of net investment is *constant* over time (the constant is not required to be zero) then this would ensure intertemporal equity (in the sense described above, but with "consumption" now interpreted as the utility based on a vector of consumption goods). Furthermore, this investment rule, which might legitimately be called the *Dixit–Hammond–Hoel rule*, was also a *necessary* condition for intertemporal equity along competitive paths.

This is an elegant characterization of competitive equitable paths. But it also implies that the special significance of Hartwick's rule for intertemporal equity should be re-examined. This question is prompted by the observation that in Solow's original exercise in the context of an exhaustible resource model, the maximin equitable paths do in fact satisfy Hartwick's rule, not just the Dixit–Hammond–Hoel rule. More recently, there has been considerable interest in this issue; see Withagen and Asheim (1998) for references to some of the literature that has emerged.

Roughly speaking, this literature might be summarized as showing that for competitive paths which are both equitable *and (long-run) efficient*,[1]

[1] In terms of the notation introduced in Section II, a path $(k(t), r(t), c(t))$ from (K, S) is called *(long-run) inefficient* if there is another path $(k'(t), r'(t), c'(t))$ from (K, S), such that $c'(t) \geq c(t)$ for $t \geq 0$, and $c'(\tau) > c(\tau)$ for some $\tau \geq 0$. It is called *(long-run) efficient* if it is not (long-run) inefficient.

Hartwick's rule must hold. In an exhaustible resource model, but without the special structure of the Cobb–Douglas technology of Solow (1974), such a result was first noted by Dasgupta and Mitra (1983). However, their treatment of equity and efficiency took place in the context of a discrete-time model, where Hartwick's rule does not hold in its original form but rather in a modified form. In the continuous-time framework of this exhaustible resource model, Hartwick's rule does hold in its original form as a necessary condition along efficient equitable paths. In fact, this result is valid in fairly general intertemporal allocation models, as demonstrated in Withagen and Asheim (1998) and Mitra (2002).[2]

The result that we prove in this paper shows that, in the context of the exhaustible resource model in which Hartwick first proposed his rule, Hartwick's rule is both necessary and sufficient for intertemporal equity of competitive paths, provided the exhaustible resource is "important" in production.[3] That is, in contrast to the literature mentioned in the preceding paragraph, the rather demanding assumption of (long-run) efficiency of these paths is irrelevant in this particular context. Our result implies the rather intriguing fact that in the context of this model, competitive paths which satisfy the Dixit–Hammond–Hoel rule (that the value of net investment be *constant*) must also satisfy Hartwick's rule (that the value of net investment be *zero*).

Our analysis also reveals a richer set of equivalence results, which may be described as follows. Consider the following three conditions that a feasible path may satisfy: (i) it is competitive; (ii) it is equitable; (iii) it satisfies Hartwick's rule. It turns out that if the path satisfies any two of these three conditions, it must also satisfy the third. In particular, this indicates that along equitable paths, Hartwick's rule ensures "myopic efficiency" (Hotelling's rule), which is quite different from the role for which it was originally introduced in the literature.

Our approach to proving the above results also appears to have novel aspects. We study in detail (in Section III) the dynamics of equitable paths, when the exhaustible resource is not necessarily "important" in production, and find that (interior) competitive equitable paths have a property intermediate between those asserted by the Dixit–Hammond–Hoel rule and Hartwick's rule. The value of net investment along such paths is a *constant* which is *non-negative*, so that capital accumulation must at least offset resource depletion in value terms at each date. When the exhaustible

[2] Our review of this literature is deliberately brief, since there is a comprehensive appraisal of this line of research in Asheim, Buchholz and Withagen (2003). In particular, their paper explores the relation between Hartwick's rule and the theory of sustainability.
[3] For a precise definition of this concept, see Section II.

resource is "important" in production, then this property can be strengthened to yield the result (in Theorem 1) that this constant cannot be positive.

II. The Framework

An Exhaustible Resource Model

Our model has one produced good, which serves as both the capital as well as the consumption good, and an exhaustible resource. Labor is assumed to be constant over time. The framework described below comprises the standard model in the literature on intertemporal resource allocation in the presence of an exhaustible resource; see, for example, Dasgupta and Heal (1979) and Solow (1974). However, it generalizes the standard model, by explicitly allowing for depreciation of augmentable capital.

Denote by k the stock of the augmentable capital good and by r the flow of the exhaustible resource used. Let $G: \mathbb{R}^2_+ \to \mathbb{R}_+$ denote the gross production function for the capital *cum* consumption good, using the capital input stock k and the exhaustible resource used, r. A function, $D: \mathbb{R}_+ \to \mathbb{R}_+$, denotes the depreciation of augmentable capital. Thus, the net production function, $F: \mathbb{R}^2_+ \to \mathbb{R}$ is defined by: $F(k, r) = G(k, r) - D(k)$ for all $(k, r) \in \mathbb{R}^2_+$. The output $G(k, r)$ can be used to replace worn-out capital, $D(k)$, to augment the capital stock through net investment, $z = \dot{k}$, or to provide consumption, c. Output $G(k, r)$ is the only source of flow of consumption or of net addition to the stock of capital.

The following assumptions are made on G:

(A.1) $G(0, r) = G(k, 0) = 0$ for $k \in \mathbb{R}_+$, $r \in \mathbb{R}_+$.

(A.2) G is continuous, concave and non-decreasing on $\mathbb{R}^2_+$, and has continuous first- and second-order partial derivatives on $\mathbb{R}^2_{++}$, with $G_1(k, r) > 0$, $G_2(k, r) > 0$.[4]

(A.3) $\beta \equiv \inf_{(k, r) \gg 0}[rG_2(k, r)/G(k, r)] > 0$.

While (A.1) and (A.2) are standard assumptions in this context, (A.3) conveys the restriction that the exhaustible resource is "important" in production; see Mitra (1978). The Cobb–Douglas production function (with capital coefficient $\alpha > 0$, resource coefficient $\beta > 0$ and $\alpha + \beta \leq 1$) satisfies (A.1)–(A.3).

[4] We are using standard notation; G_i, for $i = 1, 2$, denotes the partial derivative of the function G with respect to the ith argument.

The following assumption is made on D:

(A.4) $D(0) = 0$; D is continuous, non-decreasing on $\mathbb{R}_+$, and has continuous first- and second-order derivatives on $\mathbb{R}_{++}$.

An important class of depreciation functions that fulfill (A.4) is given by $D(k) = \delta k^\theta$ with $\delta \geq 0$ and $\theta \in [0, 1]$. The standard exponential depreciation case is obtained as a special case of (A.4) with $\delta > 0$ and $\theta = 1$. In the theory of exhaustible resources, it is common to assume that augmentable capital is non-depreciating, and this is obtained as a special case of (A.4) with $\delta = 0$. The case of $\delta > 0$ and $\theta \in [0, 1)$ provides a more realistic set-up to which the results of this paper apply.

The case of non-depreciating capital is, of course, an idealization, like a frictionless universe. The case of $\delta > 0$ and $\theta \in [0, 1)$ provides a more interesting scenario. The capital stock, k, might be viewed as an (idealized) aggregate of various vintages of machines. A higher capital stock would then be typically attained with more machines of newer vintages (which are better quality machines), leading to a lower overall depreciation rate, $[D(k)/k]$.

Competitive Paths

A *path* from (initial stocks of capital and the exhaustible resource) (K, S) in $\mathbb{R}^2_+$ is a triplet of functions $(k(t), r(t), c(t))$, where $k(\cdot) : [0, \infty) \to \mathbb{R}_+$, $r(\cdot) : [0, \infty) \to \mathbb{R}_+$ and $c(\cdot) : [0, \infty) \to \mathbb{R}_+$, such that $k(t)$, $r(t)$, $c(t)$ are continuously differentiable functions[5] of t, and satisfy:[6]

$$\left.\begin{aligned} &\text{(a) } c(t) = F(k(t), r(t)) - \dot{k}(t) \quad \text{for } t \geq 0, \\ &\text{(b) } \int_0^\infty r(t)dt \leq S, \\ &\text{(c) } k(0) = K. \end{aligned}\right\} \tag{1}$$

[5] In a continuous-time model, Hotelling's rule can only be formulated when the marginal product of the resource is a differentiable function of time. The only general way of achieving this is to assume that along the class of paths considered, $k(t)$ and $r(t)$ are differentiable functions of time, and appeal to the chain rule, using (A.2) and (A.4). Hence, in order to formulate Hartwick's rule, then, we need $\dot{k}(t)$ to be a differentiable function of time. But, then, feasibility (that is, (1)(a)) requires that $c(t)$ be a differentiable function of time.

[6] Our definition of a path allows capital equipment to be directly consumed. An alternative is to define paths in such a way as to incorporate the restriction that investment is "irreversible": $\dot{k}(t) \geq -D(k(t))$ for $t \geq 0$. The methods developed in this paper can also be applied to this alternative formulation.

A path ($k(t)$, $r(t)$, $c(t)$) from (K, S) in $\mathbb{R}^2_+$ is called *interior* if $k(t) > 0$, $r(t) > 0$ and $c(t) > 0$ for $t \geq 0$.[7]

An interior path ($k(t)$, $r(t)$, $c(t)$) from (K, S) in $\mathbb{R}^2_+$ is called *competitive* if it satisfies *Hotelling's rule*, equating the returns on the capital good and the exhaustible resource:[8]

$$\dot{F}_2(k(t), r(t))/F_2(k(t), r(t)) = F_1(k(t), r(t)). \tag{2}$$

If ($k(t)$, $r(t)$, $c(t)$) is an interior path from (K, S) in $\mathbb{R}^2_+$, then $F_2(k(t), r(t)) > 0$ for $t \geq 0$ and we can associate with it a path of prices ($p(t)$) defined as follows:

$$p(t) = 1/F_2(k(t), r(t)). \tag{3}$$

Note that if $p(\cdot) : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is any differentiable function of time, such that the following *profit-maximization* condition[9]

$$p(t)F(k(t), r(t)) - (-\dot{p}(t))k(t) - r(t) \geq p(t)F(k, r) - (-\dot{p}(t))k - r \tag{4}$$

is satisfied by the interior path ($k(t)$, $r(t)$, $c(t)$) for all $t \geq 0$, and all $(k, r) \in \mathbb{R}^2_+$, then clearly it satisfies the first-order conditions:

$$p(t)F_1(k(t), r(t)) + \dot{p}(t) = 0; \quad p(t)F_2(k(t), r(t)) = 1 \quad \text{for } t \geq 0 \tag{5}$$

so that Hotelling's rule (2) must hold.[10]

Equitable Paths

A path ($k(t)$, $r(t)$, $c(t)$) from (K, S) is called *equitable* if $c(t)$ is constant over time. An important question that arises in the context of an exhaustible resource model is whether there exist equitable paths where constant consumption is actually positive. It was first posed by Solow (1974), who answered it in the context of the class of CES production functions and non-depreciating capital. His result was subsequently generalized by several

[7] Note that along an interior path ($k(t)$, $r(t)$, $c(t)$), using (1)(a) and assumptions (A.2) and (A.4), $\dot{k}(t)$ is itself a continuously differentiable function of t.

[8] This rule incorporates what might be called *short-run* or *myopic efficiency*; see, for example, Dasgupta and Heal (1979).

[9] Here, $p(t)$ is the price of the capital *cum* consumption good, measured in units of the resource stock at time 0, which is the *numéraire*, and ($-\dot{p}(t)$) is to be interpreted as the rental rate on capital. Since the resource stock does not directly affect output at any t (it is only the flow of the resource which affects the output), the present-value price of the resource is constant; this constant is positive by assumption (A.2). Thus, the present-value price of the resource at time t, measured in units of the resource stock at time 0, is unity; see Mitra (1978) for a formal proof.

[10] Since F need not be concave, the first-order condition (2) to the maximization problem involved in (4) need not be sufficient. So the usual equivalence between profit maximization and Hotelling's rule, noted in the literature under non-depreciating capital, need not hold in our framework.

authors; a very general characterization of technologies, for which equitable paths with positive consumption will exist, was provided in Cass and Mitra (1991).

The importance of our results here depends on whether equitable paths with positive consumption exist. In order to show that our results are non-vacuous even in the case with depreciating capital, we consider a particular example whose structure, however, can be generalized.

Assume that $G(k, r) = k^{\alpha}r^{\beta}$, where $\alpha > \beta > 0, \alpha + \beta \leq 1$, and $D(k) = \delta k^{\theta}$, where $\delta \geq 0$ and $\theta \in [0, 1]$. For the zero depreciation case, Solow (1974) found the equitable path with the maximum constant consumption level (among all equitable paths). This maximum level can be solved in terms of the technological parameters and the initial conditions; see Solow (1974, p. 39). We can thus write it as a function $c(K, S, \alpha, \beta)$, where

$$c(K, S, \alpha, \beta) = [(1 - \beta)(\alpha - \beta)^{\beta/(1-\beta)}]S^{\beta/(1-\beta)}K^{(\alpha-\beta)/(1-\beta)}. \tag{6}$$

Now let $\delta > 0$ and $\theta \in [0, 1)$ be given. If

$$\alpha - \theta > \beta \tag{7}$$

then, using (6), we can find $K(\delta, \theta)$ such that:

$$c(K(\delta, \theta), S, \alpha - \theta, \beta) = \delta. \tag{8}$$

Consider now any initial stock of capital, K, such that:

$$K > \max\{1, K(\delta, \theta)\}. \tag{9}$$

As is shown in the Appendix, there is then an equitable path that has at least the consumption level $c(K, S, \alpha - \theta, \beta) - \delta > 0$.

It is worth noting that, for this existence result, two conditions are crucial. Condition (7) is a straightforward extension of the Solow condition $\alpha > \beta$. This condition is certainly violated if $\theta = 1$ and, in this case (with $\delta > 0$), it is known that there is no equitable path with positive consumption. The other condition (9) expresses a restriction on the initial capital stock. In particular, even if $\theta = 0$, so that (7) is automatically satisfied under Solow's condition $\alpha > \beta$, it is easy to check that (with $\delta > 0$) there is no equitable path with positive consumption for sufficiently low but positive initial capital stocks.

III. Preliminary Results

We now turn to some preliminary propositions, which will be useful in establishing the principal result of this paper in the next section. They are also of independent interest, since they provide a significant insight into the possible dynamics of the relevant variables in the exhaustible resource model.

A Fundamental Identity

The first result is a fundamental identity (stated below in Proposition 1 and proved in the Appendix) which relates the rate of change in consumption to the rate of change in the capital stock for *any* (interior) path. For notational ease we drop the arguments of the functions appearing below. Unless mentioned otherwise, it is understood that all functions are evaluated at time t and along the path, that is, at the point $(k(t), r(t))$ represented by $(\cdot)$.

Proposition 1. *Under (A.2) and (A.4), an interior path* $(k(t), r(t), c(t))$ *from* (K, S) *in* $\mathbb{R}^2_+$ *satisfies the following*:

$$\dot{c}(t) + F_2(\cdot)\frac{d}{dt}\left[\frac{\dot{k}(t)}{F_2(\cdot)} - r(t)\right] + \left[\frac{\dot{F}_2(\cdot)}{F_2(\cdot)} - F_1(\cdot)\right]\dot{k}(t) = 0 \quad \textit{for all } t \geq 0. \quad (10)$$

The Dixit–Hammond–Hoel Rule and Capital Accumulation

Hartwick's rule (HR) is a prescription to invest resource rents in the accumulation of the (augmentable) capital good; that is:

$$\dot{k}(t) = r(t)F_2(k(t), r(t)) \quad \text{for } t \geq 0. \quad (11)$$

Defining the time path of prices $p(t)$ as in (3) above, we can rewrite this as:

$$p(t)\dot{k}(t) - r(t) = 0 \quad \text{for } t \geq 0, \quad (12)$$

which says that the value of net investment, including changes in the stock of the resource as well as that of the augmentable capital good, is *zero* at each t.

The Dixit–Hammond–Hoel (DHH) rule is a generalized version of HR which says that the value of net investment, inclusive of changes in resource stocks is *constant*; that is, there is a number E such that:

$$p(t)\dot{k}(t) - r(t) = E \quad \text{for } t \geq 0. \quad (13)$$

Our second preliminary result (stated as Proposition 2 below, and proved in the Appendix) shows that for an (interior) equitable path, satisfying the DHH rule, there must be capital accumulation for all t. For this result, we introduce an additional assumption:

(A.5) For every $a \in \mathbb{R}_{++}$, there is $k(a) \in \mathbb{R}_{+}$, such that $[G(k, a) - D(k)]$ is increasing in k for $k > k(a)$.

If $G(k, r) = k^{\alpha}r^{\beta}$, where $\alpha > 0$, $\beta > 0$, $\alpha + \beta \leq 1$, and $D(k) = \delta k^{\theta}$, where $\delta \geq 0$ and $\theta \in [0, 1]$, then (A.5) is satisfied if $\alpha > \theta$, by defining $k(a) = [\delta\theta/\alpha a^{\beta}]^{[1/(\alpha-\theta)]}$ for each $a > 0$. When capital is non-depreciating, as is commonly assumed in the theory of exhaustible resources, (A.5) is automatically satisfied whenever G satisfies (A.2), by defining $k(a) = 0$.

Proposition 2. *(i) Under (A.1), (A.2), (A.4), if* $(k(t), r(t), c(t))$ *is an interior equitable path then*:

$$\sup_{t \geq 0} k(t) = \infty. \tag{14}$$

(ii) Under (A.1), (A.2), (A.4), (A.5), if $(k(t), r(t), c(t))$ *is an interior equitable path which satisfies the DHH rule, then*:

$$(a)\ p(t)\dot{k}(t) \geq r(t) \quad \text{for } t \geq 0;$$
$$(b)\ \dot{k}(t) > 0 \quad \text{for } t \geq 0.$$

Remark 1. Result (ii)(a) of Proposition 2 yields the result, noted in (ii)(b), that an (interior) equitable path satisfying the DHH rule must accumulate capital at all dates. But it is of independent interest, since it indicates that along such a path, capital accumulation at least offsets resource depletion in value terms. The error, if any, is in "overcompensating" for resource depletion by capital accumulation (in value terms).

An Equivalence Result

Hartwick (1977) showed that if an interior competitive path $(k(t), r(t), c(t))$ satisfies (11), then it is equitable. In a more general framework, Dixit, Hammond and Hoel (1980) have shown that an interior competitive path is equitable if and only if it satisfies (13) above.

Hartwick's result may be regarded as part of the following observation: if an interior path $(k(t), r(t), c(t))$ satisfies HR (11), then it is equitable *if and only if* it is competitive. This observation, in turn, is a special case of the following result. Consider three conditions that a feasible path may satisfy: (i) it is competitive; (ii) it is equitable; (iii) it satisfies the DHH rule. It turns out that if the path satisfies any two of these three conditions, it must also satisfy the third; see Proposition 3 below.

Proposition 3. *Under (A.1), (A.2), (A.4), (A.5), if* $(k(t), r(t), c(t))$ *is an interior path from* (K, S) *in* $\mathbb{R}^2_+$ *which satisfies any two of the following three conditions: (i) it is competitive, (ii) it is equitable and (iii) it satisfies the DHH rule, then it must also satisfy the remaining condition.*

Proof: First note that (ii) is equivalent to $\dot{c}(t) = 0$ for all $t \geq 0$; and (iii) is equivalent to:

$$\frac{d}{dt}[p(t)\dot{k}(t) - r(t)] = 0 \quad \text{for all } t \geq 0.$$

Clearly, it follows directly from (10), that (i) and (iii) imply (ii). Since $F_2(k(t), r(t)) > 0$ for all $t \geq 0$ by (A.2), it also follows from (10) that (i) and (ii) imply (iii). Finally, if (ii) and (iii) hold, then (10) implies:

$$\left[\frac{\dot{F}_2(k(t), r(t))}{F_2(k(t), r(t))} - F_1(k(t), r(t))\right]\dot{k}(t) = 0 \quad \text{for all } t \geq 0.$$

But, by Proposition 2, we also know that $\dot{k}(t) > 0$ for all $t \geq 0$. Thus, (i) must hold. ■

Remark 2. Note that, since HR is a special case of the DHH rule, it follows from Proposition 3 that an interior path $(k(t), r(t), c(t))$, which satisfies HR, is equitable if and only if the path is competitive.

IV. Equity Implies Hartwick's Rule

We are now able to establish our main result: that if an interior competitive path $(k(t), r(t), c(t))$ is equitable, then it satisfies HR (11).

Theorem 1. *Under (A.1)–(A.5), if an interior competitive path* $(k(t), r(t), c(t))$ *from* (K, S) *in* $\mathbb{R}^2_+$ *is equitable then it satisfies HR; that is, (11) holds.*

Proof: Let $(k(t), r(t), c(t))$ be an interior equitable competitive path from $(K, S) \in \mathbb{R}^2_+$, with associated prices $(p(t))$, defined as in (3). By Proposition 3, we know that the DHH rule must be satisfied. That is, there is a real number E such that:

$$p(t)\dot{k}(t) - r(t) = E \quad \text{for all } t \geq 0. \tag{15}$$

By Proposition 2, we must have $E \geq 0$. Further, for $t \geq 0$:

$$\begin{aligned}[1/\beta]r(t) &\geq [G(k(t),r(t))/r(t)G_2(k(t),r(t))]r(t) \\ &= p(t)G(k(t),r(t)) \\ &= p(t)[\dot{k}(t) + D(k(t)) + c(t)] \\ &\geq p(t)\dot{k}(t) \geq E, \end{aligned} \tag{16}$$

where the first inequality follows from assumption (A.3), and the last inequality from (15). If $E > 0$, then for $t \geq 0$, we have $r(t) \geq \beta E > 0$, which violates the resource constraint (1)(b). Thus, we must have $E = 0$, so (11) must hold. ∎

The following interesting consequence of Theorem 1 is worth recording separately.

Corollary 1. *Under (A.1)–(A.5), if an interior competitive path* ($k(t)$, $r(t)$, $c(t)$) *from* (K, S) *in* $\mathbb{R}^2_+$ *satisfies the DHH rule, then it satisfies HR.*

Proof: Under the assumptions of the corollary, it follows from Proposition 3 that the path is equitable and, hence, by Theorem 1, it satisfies HR. ∎

It is an immediate consequence of Theorem 1, and its corollary, that the set of equivalences, stated in Proposition 3 (in Section III), takes the sharper form where the DHH rule may be replaced by HR. We state this below as a proposition.

Proposition 4. *Under (A.1)–(A.5), if an interior path* ($k(t)$, $r(t)$, $c(t)$) *from* (K, S) *in* $\mathbb{R}^2_+$ *satisfies any two of the following three conditions: (i) it is competitive, (ii) it is equitable and (iii) it satisfies HR, then it must also satisfy the remaining condition.*

V. Conclusion

In this paper we have established (in Theorem 1) the converse of Hartwick's result: in a model with an exhaustible resource which is "important" in production, equitable paths which are competitive (that is, which satisfy Hotelling's rule), must satisfy Hartwick's rule. In contrast to the existing literature, we show that the necessity of Hartwick's rule for equitable paths follows from short-run or myopic efficiency (Hotelling's rule). The more demanding condition of a long-run efficiency or maximin characteristic of equitable paths is not relevant in this context.

As a consequence of this theorem, we note two other interesting results. First, in the context of our model, competitive paths which satisfy the Dixit–Hammond–Hoel rule (that the value of net investment, inclusive of changes in the stock of the exhaustible resource, be *constant*) must also satisfy

Hartwick's rule (that the value of net investment be *zero*). Second, there is a useful set of equivalence results involving equity, Hotelling's rule and Hartwick's rule: if a path satisfies any two of these three conditions then it must necessarily satisfy the remaining one (Proposition 4).

In the course of establishing these principal results of the paper, we introduce two findings (Propositions 2 and 3) which are valid even when the exhaustible resource is not assumed to be important in production, and are therefore of wider scope. Taken together, they ensure that for an (interior) equitable path which is competitive, there must be capital accumulation at all dates, and the capital accumulation must at least offset resource depletion in value terms.

It would be interesting to extend the results of this paper to a model with exogenous population growth. In Mitra (1983), it was shown (in a discrete-time setting) that in the context of Solow's model (with a Cobb–Douglas production function and non-depreciating capital), it is possible to have "quasi-arithmetic growth" of population (so that $L(t)$, the population at time t, satisfies $L(t) = L(0) + at^{\lambda}$ for $t \geq 0$, where $a > 0$ and $\lambda > 0$ are parameters) along equitable paths on which *per capita consumption* is a positive constant, provided $\lambda < (\alpha/\beta) - 1$. The relationship of equitable paths in a continuous-time version of this model to an appropriately modified form of Hartwick's rule is a topic currently under study by several researchers.

Appendix

Proof of an Equitable Path

Solow's equitable path $(k'(t), r'(t), c'(t))$ from (K, S), for the economy (without capital depreciation), described by the parameters $(K, S, \alpha', \beta) = (K, S, \alpha - \theta, \beta)$, satisfies:

$$\left.\begin{aligned} &\text{(a) } c(K,S,\alpha-\theta,\beta) = k'(t)^{\alpha-\theta} r'(t)^{\beta} - \dot{k}'(t) \quad \text{for } t \geq 0, \\ &\text{(b) } \int_0^{\infty} r'(t)dt = S, \\ &\text{(c) } k'(0) = K, \\ &\text{(d) } \dot{k}'(t) > 0 \quad \text{for } t \geq 0. \end{aligned}\right\} \tag{A1}$$

Thus, we can use (9) and (A1)(d) to observe that:

$$k'(t) \geq K > \max\{1, K(\delta,\theta)\} \quad \text{for } t \geq 0$$

and we can use (A1)(a) to write:

$$\begin{aligned} k'(t)^{\alpha} r'(t)^{\beta} - \delta k'(t)^{\theta} &= k'(t)^{\theta}[k'(t)^{\alpha-\theta} r'(t)^{\beta} - \delta] \\ &\geq [k'(t)^{\alpha-\theta} r'(t)^{\beta} - \delta] \\ &= \dot{k}'(t) + [c(K,S,\alpha-\theta,\beta) - \delta]. \end{aligned}$$

Thus, the economy (with capital depreciation), described by the parameters (K, S, α, β, δ, θ), has a path $\{k''(t), r''(t), c''(t)\}$ from (K, S) such that $(k''(t), r''(t)) = (k'(t), r'(t))$ for $t \geq 0$ and:

$$\begin{aligned} c''(t) &\equiv k'(t)^{\alpha} r'(t)^{\beta} - \delta k'(t)^{\theta} - \dot{k}'(t) \\ &\geq [c(K, S, \alpha - \theta, \beta) - \delta] > 0 \quad \text{for } t \geq 0. \end{aligned}$$

Thus, there is also an equitable path for this economy with the constant consumption at each date at least as large as $[c(K, S, \alpha - \theta, \beta) - \delta] > 0$.

Proof of Proposition 1

To see this, differentiate the feasibility condition:

$$c(t) = F(k(t), r(t)) - \dot{k}(t)$$

to get:

$$\dot{c}(t) = F_1(\cdot)\dot{k}(t) + F_2(\cdot)\dot{r}(t) - \frac{d}{dt}\dot{k}(t). \tag{A2}$$

Then noting that,

$$\frac{d}{dt}\left[\frac{\dot{k}(t)}{F_2(\cdot)} - r(t)\right] = \frac{1}{F_2(\cdot)}\frac{d}{dt}\dot{k}(t) - \dot{k}(t)\frac{1}{F_2(\cdot)^2}\dot{F}_2(\cdot) - \dot{r}(t)$$

we get:

$$F_2(\cdot)\frac{d}{dt}\left[\frac{\dot{k}(t)}{F_2(\cdot)} - r(t)\right] + \frac{\dot{F}_2(\cdot)}{F_2(\cdot)}\dot{k}(t) = \frac{d}{dt}\dot{k}(t) - F_2(\cdot)\dot{r}(t). \tag{A3}$$

Adding (A2) and (A3) yields (10) and completes the proof of the proposition.

Proof of Proposition 2

(i) If $(k(t), r(t), c(t))$ is an interior equitable path, then there is $c > 0$ such that $c(t) = c$ for $t \geq 0$. Suppose, contrary to (14), there is some real number $B > 0$, such that $k(t) \leq B$ for $t \geq 0$. Then, we have:

$$\dot{k}(t) = G(k(t), r(t)) - D(k(t)) - c \leq G(B, r(t)) - c \quad \text{for } t \geq s. \tag{A4}$$

Define $g(r) = G(B, r)$ for $r \geq 0$. Then, $g(0) = 0$, and g is continuous, increasing and concave on $\mathbb{R}_+$. Using Jensen's inequality, we have for all $T > 0$:

$$(1/T)\int_0^T g(r(t))dt \leq g\Big((1/T)\int_0^T r(t)dt\Big) \leq g(S/T). \tag{A5}$$

Using (A4) and (A5), we get:

$$k(T) - k(0) = \int_0^T \dot{k}(t)dt \leq \int_0^T g(r(t))dt - Tc \leq T[g(S/T) - c]. \tag{A6}$$

Since $g(S/T) \to 0$ as $T \to \infty$, (A6) implies that $k(T) < 0$ for large T, a contradiction. Thus, (14) must hold, and (i) is established.

(ii)(a) If $(k(t), r(t), c(t))$ is an interior equitable path which satisfies the DHH rule, then there is $c > 0$ such that $c(t) = c$ for $t \geq 0$, and there is a real number E such that (13) holds. Suppose, contrary to the proposition, that $E < 0$.

Clearly $\dot{k}(t) = 0$ for some $t \geq 0$. Otherwise, by continuity of $\dot{k}(t)$, we must have either (I) $\dot{k}(t) > 0$ for all $t \geq 0$, or (II) $\dot{k}(t) < 0$ for all $t \geq 0$. In case (I), by the DHH rule, we must have $r(t) > (-E)$ for all $t \geq 0$, which violates the resource constraint (1)(b). In case (II), we must have $k(t) \leq K$ for $t \geq 0$, which violates (14).

Consider the set:

$$H = \{k \in \mathbb{R}_+ : G(k, -E) - D(k) = c\}. \tag{A7}$$

Since $\dot{k}(t) = 0$ for some $t \geq 0$, we must have $r(t) = (-E)$ and $G(k(t), -E) - D(k(t)) = c$ for that t. Thus, H is non-empty. We now claim that H is bounded. Otherwise, there would exist a sequence $\{k_n\}_{n \in \mathbb{N}}$ with $k_n \in H$, $k_n \geq 1$ for each $n \in \mathbb{N}$[11] and:

$$k_n \uparrow \infty \quad \text{as } n \to \infty. \tag{A8}$$

Then there is $N \in \mathbb{N}$ such that $k_n > k(-E)$ for all $n \geq N$, where $k(-E)$ is given by assumption (A.5). Using (A.5), and the definition of H, we then have:

$$c = G(k_{N+1}, (-E)) - D(k_{N+1}) > G(k_N, (-E)) - D(k_N) = c, \tag{A9}$$

a contradiction, establishing our claim that H is bounded.

Denote sup H by $\tilde{k}$, and select any $\bar{k} > \max\{K, \tilde{k}\}$. Denote $\inf\{t \in \mathbb{R}_+ : k(t) \geq \bar{k}\}$ by T. By (14), T is well defined and $T > 0$. Further, by continuity of $k(t)$, we have:

$$k(T) \geq \bar{k}. \tag{A10}$$

By definition of T, we have $k(t) < \bar{k}$ for $t < T$, and so:

$$\dot{k}(T) \geq 0. \tag{A11}$$

If we have $\dot{k}(t) \geq 0$ for all $t \geq T$, then by the DHH rule, $r(t) \geq (-E) > 0$ for all $t \geq T$ and the resource constraint (1)(b) would be violated. Thus, there is some $\tau > T$ for which $\dot{k}(\tau) < 0$. Denote $\inf\{t \geq T : \dot{k}(t) \leq 0\}$ by L. Then, L is well defined, and by (A11) and the continuity of $\dot{k}(t)$,

$$\dot{k}(L) = 0, \tag{A12}$$

so that by the DHH rule, $r(L) = (-E)$, and $k(L) \in H$.

On the other hand, by the definition of L and (A10) we have $k(L) \geq k(T) \geq \bar{k} > \tilde{k}$, which contradicts the fact that $\tilde{k}$ is sup H, thereby establishing part (ii)(a) of the proposition.

(ii)(b) This follows directly from (ii)(a), since $r(t) > 0$ for all $t \geq 0$ for an interior path.

[11] $\mathbb{N}$ is the set of natural numbers $\{1, 2, 3, \ldots\}$.

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First version submitted August 2002;
final version received August 2004.